A Branch-Cut-and-Price Algorithm for the Location-Routing Problem

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Location-Routing Problem (LRP)

Data

- I — set of potential depots with opening costs f_i and capacities w_i, $i \in I$
- J — set of customers with demands d_j, $j \in J$
- Sets of edges: $E = J \times J$, $F = I \times J$
- c_e — transportation cost of edge $e \in E \cup F$
- An unlimited set of vehicles with capacity Q.

The problem

- Decide which depots to open
- Assign every client to an open depot subject to depot capacity
- For every depot, divide assigned clients into routes subject to vehicle capacity
- Minimize the total depot opening and transportation cost

LRP: an illustration

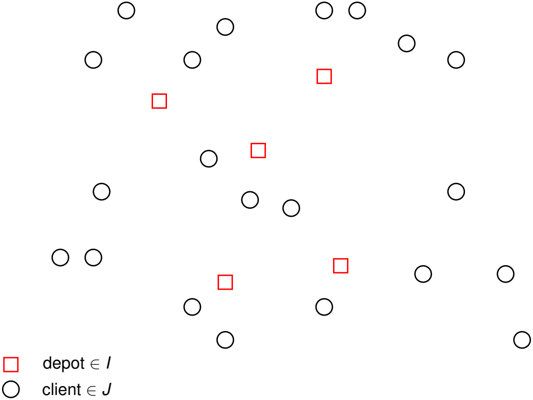

Figure: LRP instance: $G = (I \cup J, E \cup F)$

LRP: a solution

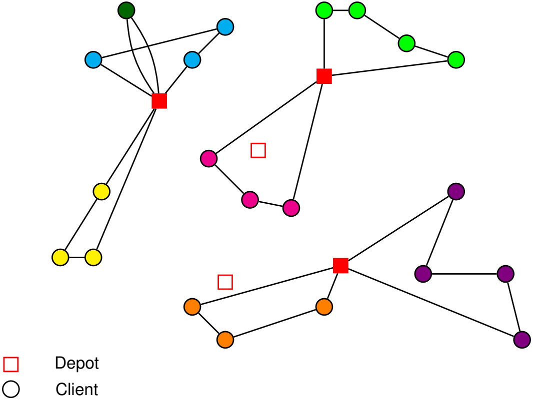

Figure: Location of depots must be jointly decided with vehicle routing.

Literature on LRP

- A combination of two central OR problems
- ≈3000 papers in Google Scholar with both “location” and “routing” in the title

Important recent works

- [Belenguer et al., 2011] — important valid inequalities & Branch-and-Cut;
- [Baldacci et al., 2011b] — exact “enumeration” & column generation approach
- [Contardo et al., 2014] — state-of-the-art exact algorithm
- [Schneider and Löffler, 2019] — state-of-the-art heuristic
- [Schneider and Drexl, 2017] — the latest survey on LRP

Our study

- Recently, large improvement in exact solution of classic VRP variants [Pecin et al., 2017b] [Pecin et al., 2017a] [S. et al., 2017] [Pessoa et al., 2018a]
- A generic Branch-Cut-and-Price VRP solver [Pessoa et al., 2019] incorporates all recent advances

  vrpsolver.math.u-bordeaux.fr
- This solver can be applied to the LRP
- However, problem-specific cuts are necessary for obtaining the state-of-the-art performance
- We review existing families of cuts and propose new ones

Formulation

- λ_r^i, $i \in I$, $r \in R_i$, equals 1 iff route r is used for depot i
- a_e^r, $e \in E \cup F$, $r \in \cup_{i \in I} R_i$, equals 1 iff edge e is used by r
- y_i, $i \in I$, equals 1 iff route depot i is open
- z_{ij}, $i \in I$, $j \in J$, equals 1 iff client j is assigned to depot i

$$\min \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{r \in R_i} \sum_{e \in E \cup F} c_e a_e^r \lambda_r^i$$

$$\begin{aligned}
\sum_{i \in I} z_{ij} &= 1, && \forall\, j \in J, \\
\sum_{r \in R_i} \sum_{e \in \delta(j)} a_e^r \lambda_r^i &= 2 z_{ij}, && \forall\, i \in I,\ j \in J \\
\sum_{j \in J} d_j z_{ij} &\leq w_i y_i, && \forall\, i \in I, \\
z_{ij} &\leq y_i, && \forall\, i \in I,\ j \in J, \\
(z, y, \lambda) &\in \{0, 1\}^K
\end{aligned}$$

Rounded Capacity Cuts [Laporte and Nobert, 1983]

Given a subset of clients $C \subset J$,

$$\sum_{i \in I} \sum_{r \in R_i} \sum_{e \in \delta(C)} a_e^r \lambda_r^i \geq 2 \cdot \left\lceil \frac{\sum_{i \in C} d_i}{Q} \right\rceil .$$

Separation (embedded in the VRP solver)

CVRPSEP library [Lysgaard et al., 2004]

Chvátal-Gomory Rank-1 Cuts [Jepsen et al., 2008] [Pecin et al., 2017c]

Each cut is obtained by a Chvátal-Gomory rounding of a set $C \subseteq J$ of set packing constraints using a vector of multipliers ρ ($0 < \rho_j < 1, j \in C$):

$$\sum_{i \in I} \sum_{r \in R_i} \left\lfloor \sum_{j \in C} \rho_j \sum_{e \in \delta(j)} \frac{1}{2} a_e^r \right\rfloor \lambda_r^i \leq \left\lfloor \sum_{j \in C} \rho_j \right\rfloor$$

All best possible vectors ρ of multipliers for $|C| \leq 5$ are given in [Pecin et al., 2017c].

Non-robust in the terminology of [Pessoa et al., 2008]

Separation (embedded in the VRP solver)

A local search for each vector of multipliers.

Depot Capacity Cuts [Belenguer et al., 2011]

If a subset of clients $C \subset J$ cannot be served by a subset of depots $S \subset I$,

$$\sum_{j \in C} d_j > \sum_{i \in S} w_i,$$

then at least one vehicle from a depot $i \in I \setminus S$ should visit C:

$$\sum_{i \in I \setminus S} \sum_{r \in R_i} \sum_{e \in \delta(C)} a_e^r \lambda_r^i \geq 2.$$

Separation (in the VRP solver callback)

A heuristic algorithm: combination of GRASP and local search.

Covering inequalities for depot capacities

Let $W = \sum_{i \in I} w_i$ and $D = \sum_{j \in J} d_j$.
We should have

$$\sum_{i \in I} w_i y_i \geq d(J) \quad \Rightarrow \quad \sum_{i \in I} w_i (1 - y_i) \leq W - D$$

We can generate any valid inequality for this knapsack. For example covering inequalities: given a subset of depots $S \subset I$, $\sum_{i \in S} w_i > W - D$,

$$\sum_{i \in S} (1 - y_i) \leq |S| - 1$$

Separation (in the VRP solver callback)

We optimize an LP which looks for the most violated inequality which is satisfied by all integer solutions of the knapsack.

Route Load Knapsack Cuts (RLKC)

x_q^i — number of routes with load of exactly $q \leq Q$ units leaving depot $i \in I$. Then:

$$\sum_{q=1}^{Q} q x_q^i \leq w_i. \tag{1}$$

Any valid inequality for (1) is valid for the LRP.

Non-robust in the terminology of [Pessoa et al., 2008]

First separation algorithm

Chvátal-Gomory rounding of (1).

$1/k$-facets of the master knapsack polytope

Theorem ([Aráoz, 1974])

The coefficient vectors ξ of the knapsack (non-trivial) facets $\xi x \leq 1$ of $\sum_{q=1}^{n} q x_q = n$ with $\xi_1 = 0$, $\xi_Q = 1$ are the extreme points of the following system of linear constraints

$$\begin{aligned} \xi_1 = 0, \quad \xi_Q &= 1, \\ \xi_q + \xi_{Q-q} &= 1 && \forall 1 \leq i \leq n/2, \\ \xi_q + \xi_t &\leq \xi_{q+t} && \textit{whenever } q + t < n. \end{aligned}$$

Definition

A knapsack facet $\xi x \leq 1$ is called a $1/k$-facet if k is the smallest possible integer such that

$$\xi_q \in \{0/k, 1/k, 2/k, \ldots, k/k\} \cup \{1/2\}.$$

Second separation algorithm

$1/6$- and $1/8$-facets can be efficiently separated using the algorithm by [Chopra et al., 2019]

Taking into account of RLKCs in the pricing

- Let $\bar{\mu}(q)$ be the contribution of RLKCs to the reduced cost of a route variable with load q
- Pricing problem: Resource Constrained Shortest Path
- It is solved by a labelling algorithm, each label L is $(\bar{c}^L + \bar{\mu}(q^L), j^L, q^L)$
- Dominance relation

$$L \succ L' \quad \text{if } \bar{c}^L \leq \bar{c}^{L'},\ j^L = j^{L'},\ q^L \leq q^{L'} \tag{2}$$

  is valid, as $\bar{\mu}(q)$ is non-decreasing
- Completion bounds can still be efficiently used as $\bar{\mu}(q)$ is super-additive

Other components of the Branch-Cut-and-Price

- Bucket graph-based labelling algorithm for the RCSP pricing [Righini and Salani, 2006] [S. et al., 2017]
- Partially elementary path (*ng*-path) relaxation [Baldacci et al., 2011a]
- Automatic dual price smoothing stabilization [Wentges, 1997] [Pessoa et al., 2018b]
- Reduced cost fixing of (bucket) arcs in the pricing problem [Ibaraki and Nakamura, 1994] [Irnich et al., 2010] [S. et al., 2017]
- Enumeration of elementary routes [Baldacci et al., 2008]
- Multi-phase strong branching [Pecin et al., 2017b]
  - On depot openings (largest priority)
  - On number of vehicles for each depot
  - On number of clients per depot
  - On assignment of clients to depots
  - On edges of the graph

Computational results

Open instances solved to optimality. Could not be solved by the state-of-the-art [Contardo et al., 2014] in 5-97 hours

Set	Instance	Optimum	Time
[Prins et al., 2006]	100x5-1b	213568	10m05s
	100x10-1a	287661	1h32m
	100x10-1b	230989	1h38m
	100x10-3a	250882	1h17m
	100x10-3b	203114	11h01m
	200x10-1a	<u>474702</u>	20m42s
	200x10-1b	375177	1h55m
	200x10-2a	<u>448005</u>	4h45m
	200x10-2b	373696	5h53m
[Tuzun and Burke, 1999]	P113112	1238.24	2h29m
	P131112	1892.17	36m52s
	P131212	1960.02	34m59s

<u>Underlined</u>: improved solutions over [Schneider and Löffler, 2019]

Sensitivity analysis of cuts specific to LRP

26 instances by [Prins et al., 2006] with 5-10 depots and 50-200 clients. Time limit 3 hours

Configuration	Solved	Root gap	Nodes	Time
All but DCCs	22/26	0.87%	27.4	611
All but RLKCs	22/26	0.51%	10.5	480
All but y-knapsack	21/26	0.69%	12.3	578
All cuts	22/26	0.47%	9.9	521

Conclusions

- A large improvement over the state-of-the-art for the LRP by applying the VRP solver and providing callbacks for problem-specific cuts
- Route Load Knapsack Cuts reduce the gap but not yet worth to include in the VRP solver
- An extension to the Two-Echelon Capacitated Vehicle Routing problem allows us to double the size of instances which can be solved to optimality [Marques et al., 2019]
- 2E-CVRP demo is available on

  vrpsolver.math.u-bordeaux.fr

Perspectives

- Improve separation of Route Load Knapsack Cuts
- A polyhedral study is needed for the Multi Capacitated Depot Vehicle Routing Problem.
- You can use the VRP solver to test new families of cuts for vehicle routing problems within state-of-the-art Branch-Cut-and-Price!

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